Faithful and deterministic teleportation of an arbitrary N-qubit state using and identifying only Bell states

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First, I show explicitly a scheme to *faithfully* and *deterministically* teleport an arbitrary 2-qubit state from Alice to Bob. In this scheme two same Bell states are sufficient for use. Bob can recover the 2-qubit state by performing at most 4 single-qubit unitary operations conditioned on Alice's 4-bit classical public message corresponding to her two Bell-state measurement outcomes. Then I generalize the 2-qubit teleportation scheme to an aribitrary $N(N \geq 3)$-qubit state teleportation case by using N same Bell states. In the generalized scheme, Alice only needs to identify N Bell states after quantum swapping and then publish her measurement outcomes ($2N$-bit classical message). Conditioned on Alice's $2N$-bit classical message, Bob only needs to perform at most $2N$ single-qubit unitary operations to *fully* recover the arbitrary state. By comparing with the newest relevant work [Phys. Rev. A**71**, 032303(2005)], the advantages of the present schemes are revealed, respectively.

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1. Introduction

Since no-cloning theorem forbids a copy of an arbitrary unknown quantum state, how to interchange different resources has ever been a question in quantum computation and quantum information. In 1993, Bennett *et al*[1] first proposed a method teleporting an arbitrary unknown quantum state in a qubit to a distant qubit with the aid of Einstein-Podlsky-Rosen (EPR) pair. Their work showed in essence the interchangeability of different resources in quantum mechanics. Hence, after Bennett *et al*'s poineering work[1], quantum teleportation has attracted many attentions in both theoretical and experimental aspects[2-18].

In theoretical aspect, one important question was whether it was possible to teleport not just a single qubit, but rather $N(N \geq 2)$ qubits. To our knowledge, so far there have been several explorations. For examples, Gao *et al* used a kind of generalized Bell states to realize a *probabilistic* two-qubit teleportation [15]; Fang *et al* presented a scheme which can *probabilistically* teleport a three-particle state via three pairs of entangled particles [12]; Yang and Guo proposed a quantum teleportation scheme of a *special* Greenberger-Horne-Rosen (GHZ) state [6]; In [8] Lee *et al* presented a scheme to teleport a *special* classes of two-qubit states; and so on. Later, Lee *et al* in a very interesting work, showed that it was possible to

teleport an *arbitrary* two-qubit state from Alice to Bob by *using a four-entangled state* and sending to him four bits of classical information [9]. Due to Lee *et al*'s inexplicit expression of their latter scheme, very recently Rigolin has revisited the scheme and generalized it to a $N(N \geq 3)$-qubit case[16]. In his work, Rigolin has explicitly presented the *faithful and deterministic* quantum teleportation scheme of an *arbitrary* two-qubit state and established its relation to multipartite entanglement. This is an important and interesting progress.

However, I think, there lie two disadvantages in Rigolin's this work[16]: (1) A complicated entangled state (e.g., one of the so-called generalized Bell states in his equations 4-19) is necessary for use. Although according to the present-day technologies it is possible to synthesize a generalized Bell state [16] or a multi-photon GHZ state from Bell states [19], the total production efficiency is lower for the produce is nondeterministic. Moreover, in fact an experimental synthesization is very difficult, especially when the photon amount in a generalized Bell state or a GHZ state is large. Incidentally, to my knowledge, so far only up to five-photon GHZ state has been synthesized[19]. Hence, it will be very nice if one can eliminate the synthesization process, i.e., directly using the same amount of Bell state instead of a generalized Bell states synthesized from these Bell states, for in this case the use of the entanglement resource is more economical and corresponding experimental prepare difficulties can be greatly reduced. (2) All the generalized Bell states must be successfully discriminated. It is generally admitted that an identification of a Bell state should be much easier than an identification of a generalized Bell state or a GHZ state, though a complete recognition of Bell states is far to come. Hence, it will also be very nice if one only needs to discriminate Bell states instead of generalized Bell states or GHZ states. In this case, obviously, the experimental discrimination difficulty is reduced. To eliminate the two disadvantages, in this paper I will present an arbitrary $N(N \geq 2)$-qubit state quantum teleportation scheme by using and identifying only Bell states. Within my scheme an *arbitrary* $N(N \geq 2)$-qubit state can also be *faithfully* and *deterministically* teleported from Alice to Bob.

The paper is organized as follows. In section 2, I will present a specific quantum teleportation scheme of an arbitrary 2-qubit state, and a comparison between this scheme and the counterpart scheme in [16] is made. Then in section 3, I will generalize the 2-qubit teleportation scheme to a $N(N \geq 3)$-qubit teleportation case also by using and identifying only Bell states. A concluding summary is given in section 4.

2. Quantum teleportation scheme of an arbitrary 2-qubit state

Before giving my specific quantum teleportation scheme of an arbitrary 2-qubit state, let us briefly review the quantum teleportation scheme of an arbitrary single-qubit state, which is originally proposed by Bennett et al in 1993[1]. The scheme is executed as follows. Alice and Bob initially share a Bell state (i.e., a maximally entangled two-qubit state), say, $|\Psi^-\rangle$. By the way, the four Bell states are defined as follows

$$|\Phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}, \tag{1}$$

$$|\Psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}. \tag{2}$$

Then the state of the joint system, i.e., the entity of the arbitrary state $|u\rangle = \alpha|0\rangle + \beta|1\rangle$ to be teleported

and the shared Bell state, can be written as

$$|\varphi\rangle = |u\rangle_x \otimes |\Psi^-\rangle_{ab}, \tag{3}$$

where x stands for the arbitrary qubit and the subscripts a and b in the Bell state $|\Psi^-\rangle$ label the qubits belonging to Alice and Bob, respectively. The equation 3 can be rewritten as follow,

$$\begin{aligned} |\varphi\rangle = \frac{1}{2}\{&|\Psi^-\rangle_{xa}(-\alpha|0\rangle_b - \beta|1\rangle_b) + |\Psi^+\rangle_{xa}(-\alpha|0\rangle_b + \beta|1\rangle_b) \\ &+|\Phi^-\rangle_{xa}(\alpha|1\rangle_b + \beta|0\rangle_b) + |\Phi^+\rangle_{xa}(\alpha|1\rangle_b - \beta|0\rangle_b)\}. \end{aligned} \tag{4}$$

Alice now performs a Bell state measurement on her two qubits and classically communicates the measurement outcome to Bob. Conditioned on Alice's two-bit information, Bob can carry out the appropriate unitary operation to reconstruct the unknown arbitrary state in his qubit b. See Table 1.

Table 1 Alice's Bell-state measurement outcome, Bob's corresponding unitary operation and the final state in his qubit. I is an identity operator and σ are the usual Pauli matrices. $\sigma^z|1\rangle = |1\rangle$, $\sigma^z|0\rangle = -|0\rangle$; $\sigma^x|1\rangle = |0\rangle$, $\sigma^x|0\rangle = |1\rangle$.

Alice's Bell-state measurement outcome	Bob's corresponding unitary operation	The final state in Bob's qubit
$\|\Psi^-\rangle_{xa}$	I_b	$-\alpha\|0\rangle_b - \beta\|1\rangle_b = -\|u\rangle_b$
$\|\Psi^+\rangle_{xa}$	σ_b^z	$\alpha\|0\rangle_b + \beta\|1\rangle_b = \|u\rangle_b$
$\|\Phi^-\rangle_{xa}$	σ_b^x	$\alpha\|0\rangle_b + \beta\|1\rangle_b = \|u\rangle_b$
$\|\Phi^+\rangle_{xa}$	$i\sigma_b^y = \sigma_b^z\sigma_b^x$	$-\alpha\|0\rangle_b - \beta\|1\rangle_b = -\|u\rangle_b$

Now let us first present our specific scheme of teleporting two arbitrary qubits by using and identifying only Bell states. Suppose that the arbitrary 2-qubit state Alice wants to teleport to Bob is written as

$$|\phi\rangle_{x_1x_2} = \alpha|0\rangle_{x_1}|0\rangle_{x_2} + \beta|0\rangle_{x_1}|1\rangle_{x_2} + \gamma|1\rangle_{x_1}|0\rangle_{x_2} + \delta|1\rangle_{x_1}|1\rangle_{x_2}, \tag{5}$$

where x_1 and x_2 label the two arbitrary qubits in Alice's site, α,β,γ and δ are *unknown* complex coefficients and $|\phi\rangle_{x_1x_2}$ is assumed to be normalized. Alice and Bob must share in advance a pair of same Bell states, say, $\Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1}$. Incidentally, the two qubits b_1 and b_2 in Bob's site are used to "receive" the teleported state from Alice. Hence, the initial joint state is

$$|\psi\rangle_{x_1x_2a_2b_2a_1b_1} = |\phi\rangle_{x_1x_2} \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1}. \tag{6}$$

It can be rewritten as

$$\begin{aligned} &|\psi\rangle_{x_1x_2a_2b_2a_1b_1} \\ =\ &[|0\rangle_{x_1}(\alpha|0\rangle_{x_2} + \beta|1\rangle_{x_2}) + |1\rangle_{x_1}(\gamma|0\rangle_{x_2} + \delta|1\rangle_{x_2})] \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1} \\ =\ &\frac{1}{2}\{|0\rangle_{x_1}[-|\Psi^-\rangle_{x_2a_2}I(\alpha|0\rangle_{b_2} + \beta|1\rangle_{b_2}) + |\Psi^+\rangle_{x_2a_2}\sigma^z_{b_2}(\alpha|0\rangle_{b_2} + \beta|1\rangle_{b_2}) \end{aligned}$$

$$
\begin{aligned}
&+|\Phi^-\rangle_{x_2a_2}I(\alpha|1\rangle_{b_2}+\beta|0\rangle_{b_2})+|\Phi^+\rangle_{x_2a_2}\sigma^x_{b_2}\sigma^z_{b_2}(\alpha|0\rangle_{b_2}+\beta|0\rangle_{b_2})] \\
+&\ |1\rangle_{x_1}[-|\Psi^-\rangle_{x_2a_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})+|\Psi^+\rangle_{x_2a_2}\sigma^z_{b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Phi^-\rangle_{x_2a_2}I(\gamma|1\rangle_{b_2}+\delta|0\rangle_{b_2})+|\Phi^+\rangle_{x_2a_2}\sigma^x_{b_2}\sigma^z_{b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})]\}\otimes\Psi^-_{a_1b_1} \\
=&\ \frac{1}{2}\{|0\rangle_{x_1}[(-|\Psi^-\rangle_{x_2a_2}I+|\Psi^+\rangle_{x_2a_2}\sigma^z_{b_2}+|\Phi^-\rangle_{x_2a_2}I+|\Phi^+\rangle_{x_2a_2}\sigma^x_{b_2}\sigma^z_{b_2})(\alpha|0\rangle_{b_2}+\beta|0\rangle_{b_2})] \\
+&\ |1\rangle_{x_1}[(-|\Psi^-\rangle_{x_2a_2}+|\Psi^+\rangle_{x_2a_2}\sigma^z_{b_2}+|\Phi^-\rangle_{x_2a_2}I+|\Phi^+\rangle_{x_2a_2}\sigma^x_{b_2}\sigma^z_{b_2})(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})]\}\otimes\Psi^-_{a_1b_1}. \quad (7)
\end{aligned}
$$

Define that

$$
O_{x_2a_2;b_2}\equiv|\Psi^-\rangle_{x_2a_2}I+|\Psi^+\rangle_{x_2a_2}\sigma^z_{b_2}+|\Phi^-\rangle_{x_2a_2}\sigma^x_{b_2}+|\Phi^+\rangle_{x_2a_2}\sigma^x_{b_2}\sigma^z_{b_2}, \quad (8)
$$

then the equation 7 can be written as

$$
\begin{aligned}
&|\psi\rangle_{x_1x_2a_2b_2a_1b_1} \\
=&\ \frac{1}{2}[|0\rangle_{x_1}O_{x_2a_2;b_2}(\alpha|0\rangle_{b_2}+\beta|0\rangle_{b_2})+|1\rangle_{x_1}O_{x_2a_2;b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})]\otimes\Psi^-_{a_1b_1} \\
=&\ \frac{1}{4}\{|\Psi^-\rangle_{x_1a_1}[-|0\rangle_{b_1}O_{x_2a_2;b_2}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})-|1\rangle_{b_1}O_{x_2a_2;b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Psi^+\rangle_{x_1a_1}[-|0\rangle_{b_1}O_{x_2a_2;b_2}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})+|1\rangle_{b_1}O_{x_2a_2;b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Phi^-\rangle_{x_1a_1}[|1\rangle_{b_1}O_{x_2a_2;b_2}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})+|0\rangle_{b_1}O_{x_2a_2;b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Phi^-\rangle_{x_1a_1}[|1\rangle_{b_1}O_{x_2a_2;b_2}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})-|0\rangle_{b_1}O_{x_2a_2;b_2}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})]\} \\
=&\ \frac{1}{4}\{-|\Psi^-\rangle_{x_1a_1}O_{x_2a_2;b_2}[|0\rangle_{b_1}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})+|1\rangle_{b_1}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Psi^+\rangle_{x_1a_1}O_{x_2a_2;b_2}\sigma^z_{b_1}[|0\rangle_{b_1}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})+|1\rangle_{b_1}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Phi^-\rangle_{x_1a_1}O_{x_2a_2;b_2}\sigma^x_{b_1}[|0\rangle_{b_1}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})+|1\rangle_{b_1}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})] \\
&+|\Phi^-\rangle_{x_1a_1}O_{x_2a_2;b_2}\sigma^x_{b_1}\sigma^z_{b_1}[|0\rangle_{b_1}(\alpha|0\rangle_{b_2}+\beta|1\rangle_{b_2})-|1\rangle_{b_1}(\gamma|0\rangle_{b_2}+\delta|1\rangle_{b_2})]\} \\
=&\ \frac{1}{4}(-|\Psi^-\rangle_{x_1a_1}O_{x_2a_2;b_2}+|\Psi^+\rangle_{x_1a_1}O_{x_2a_2;b_2}\sigma^z_{b_1}+|\Phi^-\rangle_{x_1a_1}O_{x_2a_2;b_2}\sigma^x_{b_1} \\
&+|\Phi^-\rangle_{x_1a_1}O_{x_2a_2;b_2}\sigma^x_{b_1}\sigma^z_{b_1})|\phi\rangle_{b_1b_2}. \quad (9)
\end{aligned}
$$

Alice performs Bell-state measurements on the qubit pairs (x_1, a_1) and (x_2, a_2) in her site. With equal probabilities (1/16) she obtains a Bell-state pair (see the equation 9 and Table 2). Then she sends to Bob a classical message of four bits (each Bell state corresponds to a classical message of two bits) to inform him her measurement outcomes. With this information Bob knows what single-qubit unitary operations (see Table 2) he must apply on his two qubits b_1 and b_2 to recover correctly the teleported state $|\phi\rangle$. After Bob's unitary operations the scheme is completed and Alice's arbitrary two-qubit state has been successfully teleported to Bob.

Let us simply compare the present scheme with the counterpart scheme in Ref.[16] as follows. (1) In the present 2-qubit quantum teleportation scheme only two same Bell states (e.g., $\Psi^-\otimes\Psi^-$) are sufficient for use, while in the 2-qubit quantum teleportation scheme in Ref.[16] a generalized Bell state (e.g., $|g_1\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$) should be prepared for use. Two Bell states can not be deterministically synthesized to a generalized Bell state, hence a synthesization of a generalized Bell state should need more than 2 Bell states on average. Moreover, if two Bell states instead of a generalized Bell state are used, then the experimental synthesization process is not

needed anymore. Obviously, in both the aspect of the entanglement resource use and the aspect of experiment, it is economical to use two Bell states directly instead of a generalized Bell state. (2) In the present 2-qubit quantum teleportation scheme only the four Bell states defined in eqs.(1-2) need to be discriminated. In contrast, within the counterpart scheme in Rigolin's work [16] all 16 generalized Bell states defined by the eqs.(4-19) in Ref.[16] should be discriminated. Obviously, the experimental recognition difficulties in the present scheme are considerably reduced. (3) In both schemes, Alice sends a 4-bit classical message corresponding to the measurement outcome(s) to Bob. (4)In both schemes, conditioned on Alice's message, Bob needs to perform at most 4 single-qubit unitary operations to fully recover the arbitrary state. Incidentally the controlled-NOT gate is also not necessary in this scheme as well as in [16]. Hence, it is obvious that the present scheme overwhelms the counterpart scheme in [16].

Table 2 Alice's Bell-state measurement outcomes and their probabilities, and Bob's corresponding unitary operations conditioned on Alice's measurement outcomes.

Alice's Bell-state measurement outcomes	probability	Bob's corresponding unitary operations
$\vert\Psi^-\rangle_{x_2a_2}, \vert\Psi^-\rangle_{x_1a_1}$	1/16	$I^{-1} = I$
$\vert\Psi^-\rangle_{x_2a_2}, \vert\Psi^+\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_1})^{-1} = \sigma^z_{b_1}$
$\vert\Psi^-\rangle_{x_2a_2}, \vert\Phi^-\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_1})^{-1} = \sigma^x_{b_1}$
$\vert\Psi^-\rangle_{x_2a_2}, \vert\Phi^+\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_1}\sigma^x_{b_1})^{-1} = \sigma^x_{b_1}\sigma^z_{b_1}$
$\vert\Psi^+\rangle_{x_2a_2}, \vert\Psi^-\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_2})^{-1} = \sigma^z_{b_2}$
$\vert\Psi^+\rangle_{x_2a_2}, \vert\Psi^+\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_2}\sigma^z_{b_1})^{-1} = \sigma^z_{b_1}\sigma^z_{b_2}$
$\vert\Psi^+\rangle_{x_2a_2}, \vert\Phi^-\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_2})^{-1} = \sigma^z_{b_2}$
$\vert\Psi^+\rangle_{x_2a_2}, \vert\Phi^+\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_2}\sigma^z_{b_1}\sigma^x_{b_1})^{-1} = \sigma^x_{b_1}\sigma^z_{b_1}\sigma^z_{b_2}$
$\vert\Phi^-\rangle_{x_2a_2}, \vert\Psi^-\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2}\sigma^x_{b_1})^{-1} = \sigma^x_{b_1}\sigma^x_{b_2}$
$\vert\Phi^-\rangle_{x_2a_2}, \vert\Psi^+\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2}\sigma^x_{b_1}\sigma^z_{b_1})^{-1} = \sigma^z_{b_1}\sigma^x_{b_1}\sigma^x_{b_2}$
$\vert\Phi^-\rangle_{x_2a_2}, \vert\Phi^-\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2})^{-1} = \sigma^x_{b_2}$
$\vert\Phi^-\rangle_{x_2a_2}, \vert\Phi^+\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2}\sigma^z_{b_1})^{-1} = \sigma^z_{b_1}\sigma^x_{b_2}$
$\vert\Phi^+\rangle_{x_2a_2}, \vert\Psi^-\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2}\sigma^z_{b_2}\sigma^x_{b_1})^{-1} = \sigma^x_{b_1}\sigma^z_{b_2}\sigma^x_{b_2}$
$\vert\Phi^+\rangle_{x_2a_2}, \vert\Psi^+\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2}\sigma^z_{b_2}\sigma^x_{b_1}\sigma^z_{b_1})^{-1} = \sigma^z_{b_1}\sigma^x_{b_1}\sigma^z_{b_2}\sigma^x_{b_2}$
$\vert\Phi^+\rangle_{x_2a_2}, \vert\Phi^-\rangle_{x_1a_1}$	1/16	$(\sigma^x_{b_2}\sigma^z_{b_2})^{-1} = \sigma^z_{b_2}\sigma^x_{b_2}$
$\vert\Phi^+\rangle_{x_2a_2}, \vert\Phi^+\rangle_{x_1a_1}$	1/16	$(\sigma^z_{b_2}\sigma^x_{b_2}\sigma^z_{b_1})^{-1} = \sigma^z_{b_1}\sigma^x_{b_2}\sigma^z_{b_2}$

3. Quantum teleportation scheme of an arbitrary $N(N \geq 3)$-qubit state

Now let us generalize the arbitrary 2-qubit state quantum teleportation scheme to an arbitrary $N(N \geq 3)$-qubit quantum state teleportation scheme. In the following I will prove that, if an arbitrary $(N-1)(N \geq 3)$-qubit quantum state can be teleported successfully between Alice and Bob via sharing $N-1$ same Bell states, identifying $N-1$ Bell states after quantum swapping, sending $2(N-1)$ bits information and performing at most $2(N-1)$ single-qubit operations, then an arbitrary $N(N \geq 3)$-qubit quantum

state can be teleported successfully via sharing N same Bell states, identifying N Bell states after quantum swapping, sending $2N$ bits information and performing at most $2N$ single-qubit operations.

Suppose that the arbitrary $n(n \geq 3)$-qubit state Alice wants to teleport to Bob is written as

$$|\xi\rangle_{x_1x_2\ldots x_N} = \sum_{m_N=0}^{1} \cdots \sum_{m_2=0}^{1} \sum_{m_1=0}^{1} C_{m_1m_2\ldots m_N} |m_1\rangle_{x_1} |m_2\rangle_{x_2} \ldots |m_N\rangle_{x_N}, \tag{10}$$

where C's are complex coefficients and $|\xi\rangle_{x_1x_2\ldots x_N}$ is assumed to be normalized. It can be decomposed as

$$\begin{aligned} |\xi\rangle_{x_1x_2\ldots x_N} &= |0\rangle_{x_N} (\sum_{m_{N-1}=0}^{1} \cdots \sum_{m_2=0}^{1} \sum_{m_1=0}^{1} C_{m_1m_2\ldots m_{N-1}0} |m_1\rangle_{x_1} |m_2\rangle_{x_2} \ldots |m_{N-1}\rangle_{x_{N-1}}) \\ &+ |1\rangle_{x_N} (\sum_{m_{N-1}=0}^{1} \cdots \sum_{m_2=0}^{1} \sum_{m_1=0}^{1} C_{m_1m_2\ldots m_{N-1}1} |m_1\rangle_{x_1} |m_2\rangle_{x_2} \ldots |m_{N-1}\rangle_{x_{N-1}}) \\ &\equiv |0\rangle_{x_N} \zeta_{x_1x_2\ldots x_{N-1}} + |1\rangle_{x_N} \zeta'_{x_1x_2\ldots x_{N-1}}. \end{aligned} \tag{11}$$

Here $\zeta_{x_1x_2\ldots x_{N-1}}$ and $\zeta'_{x_1x_2\ldots x_{N-1}}$ are in essence arbitrary $(N-1)$-qubit states, respectively. Note that they have the same form except for their coefficients. This is important for the later use.

Alice and Bob must share in advance N same Bell states, say, $\Psi^-_{a_Nb_N} \otimes \ldots \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1}$. Similarly, as mentioned before, the N qubits b_1, b_2, $\ldots$, $b_{N-1}$and b_N in Bob's site are used to "receive" the teleported state from Alice. Hence, the initial joint state is

$$\begin{aligned} & |\chi\rangle_{x_1x_2\ldots x_Na_Nb_N\ldots a_2b_2a_1b_1} \\ = \; & |\xi\rangle_{x_1x_2\ldots x_N} \otimes \Psi^-_{a_Nb_N} \otimes \ldots \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1} \\ = \; & [|0\rangle_{x_N} (\zeta_{x_1x_2\ldots x_{N-1}} \otimes \Psi^-_{a_{N-1}b_{N-1}} \otimes \ldots \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1}) \\ + \; & |1\rangle_{x_N} (\zeta'_{x_1x_2\ldots x_{N-1}} \otimes \Psi^-_{a_{N-1}b_{N-1}} \otimes \ldots \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1})] \otimes \Psi^-_{a_Nb_N}. \end{aligned} \tag{12}$$

Since I have previously supposed that an arbitrary $(N-1)(N \geq 3)$-qubit quantum state can be teleported successfully via using and identifying only Bell states, then the following equation holds,

$$\begin{aligned} & \zeta_{x_1x_2\ldots x_{N-1}} \otimes \Psi^-_{a_{N-1}b_{N-1}} \otimes \ldots \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1} \\ = \; & \frac{1}{2^{N-1}} \sum_{i=1}^{2^{N-1}} \mathcal{H}_{i;x_{N-1}a_{N-1}} \ldots \mathcal{P}_{i;x_2a_2} \mathcal{Q}_{i;x_1a_1} U_{i;b_{N-1}\ldots b_2b_1} \zeta_{b_1b_2\ldots b_{N-1}} \\ \equiv \; & \frac{1}{2^{N-1}} \sum_{i=1}^{2^{N-1}} O_i \zeta_{b_1b_2\ldots b_{N-1}}, \end{aligned} \tag{13}$$

where each calligraphic letter stands for a Bell state and U is a unitary operation containing at most $2(N-1)$ single-qubit operations, hence each O contains $N-1$ Bell states and at most $2(N-1)$ single-qubit operations. By the way, since $\zeta_{x_1x_2\ldots x_{N-1}}$ and $\zeta'_{x_1x_2\ldots x_{N-1}}$ have the same form but their coefficients, the equation 13 also holds for the prime case, that is,

$$\zeta'_{x_1x_2\ldots x_{N-1}} \otimes \Psi^-_{a_{N-1}b_{N-1}} \otimes \ldots \otimes \Psi^-_{a_2b_2} \otimes \Psi^-_{a_1b_1} = \frac{1}{2^{N-1}} \sum_{i=1}^{2^{N-1}} O_i \zeta'_{b_1b_2\ldots b_{N-1}}, \tag{14}$$

Then the equation 12 can be written as

$$
\begin{aligned}
& |\chi\rangle_{x_1 x_2 \ldots x_N a_N b_N \ldots a_2 b_2 a_1 b_1} \\
= & \frac{1}{2^{N-1}} \sum_{i=1}^{2^{N-1}} (|0\rangle_{x_N} O_i \zeta_{b_1 b_2 \ldots b_{N-1}} + |1\rangle_{x_N} O_i \zeta'_{b_1 b_2 \ldots b_{N-1}}) \otimes \Psi^-_{a_N b_N} \\
= & \frac{1}{2^N} \sum_{i=1}^{2^{N-1}} [-|\Psi^-\rangle_{x_N a_N} (O_i \zeta_{b_1 b_2 \ldots b_{N-1}} |0\rangle_{b_N} + O_i \zeta'_{b_1 b_2 \ldots b_{N-1}} |1\rangle_{b_N}) \\
& + |\Psi^+\rangle_{x_N a_N} (-O_i \zeta_{b_1 b_2 \ldots b_{N-1}} |0\rangle_{b_N} + O_i \zeta'_{b_1 b_2 \ldots b_{N-1}} |1\rangle_{b_N}) \\
& + |\Phi^-\rangle_{x_N a_N} (O_i \zeta_{b_1 b_2 \ldots b_{N-1}} |1\rangle_{b_N} + O_i \zeta'_{b_1 b_2 \ldots b_{N-1}} |0\rangle_{b_N}) \\
& + |\Phi^+\rangle_{x_N a_N} (O_i \zeta_{b_1 b_2 \ldots b_{N-1}} |1\rangle_{b_N} - O_i \zeta'_{b_1 b_2 \ldots b_{N-1}} |0\rangle_{b_N})] \\
= & \frac{1}{2^N} \sum_{i=1}^{2^{N-1}} (-|\Psi^-\rangle_{x_N a_N} O_i + |\Psi^+\rangle_{x_N a_N} O_i \sigma^z_{b_N} + |\Phi^-\rangle_{x_N a_N} O_i \sigma^x_{b_N} \\
& + |\Phi^-\rangle_{x_N a_N} O_i \sigma^x_{b_N} \sigma^z_{b_N}) |\xi\rangle_{b_1 b_2 \ldots b_N}. \qquad (15)
\end{aligned}
$$

The equation 15 has shown that, if Alice performs N Bell state measurements and publishes $2N$ bits information, then conditioned on Alice's information, Bob can recover the arbitrary state $|\xi\rangle$ by performing at most $2N$ single-qubit operations. As a matter of fact, in section 2 I have already shown that any arbitrary 2-qubit state can be teleported by using and identifying only Bell states, hence, in terms of recurrence it is easily concluded that any arbitrary $N(N \geq 3)$-qubit state can also be successfully teleported by using and identifying only Bell states. So far I have generalized the previous specific 2-qubit quantum state teleportation scheme to a N-qubit case.

Now let us simply compare the present scheme with the counterpart scheme in Ref.[16] as follows: (1) In the scheme in Ref.[16], Alice needs to share a $2N$-qubit generalized Bell states with Bob, while in this scheme Alice only needs to share N same Bell states with Bob. (2) In the scheme in Ref.[16], Alice needs to realize a $2N$-qubit generalized Bell state measurement. In the present scheme, only N Bell states need to be identified. (3) Alice needs to publish a messsge of $2N$ classical bits corresponding to her $2N$-qubit generalized Bell state measurement outcome. In the present scheme, Alice also needs to publish a messsge of $2N$ classical bits corresponding to her N Bell state measurement outcomes. (4) In both scheme, Bob needs to perform at most $2N$ single-qubit operations conditioned on Alice's message. Obviously, the present scheme is preponderant.

4. Summary

I have explicitly shown a teleportation scheme that allows Alice to faithfully and deterministically teleport an arbitrary 2-qubit state to Bob. In the scheme, only two same Bell states need to be shared by Alice and Bob. Only four Bell states must be discriminated by Alice. Bob needs to perform at most 4 single-qubit operations conditioned on Alice's 4-bit classical message. I have generalized this scheme to a $N(N \geq 3)$-qubit teleportation case, where only N same Bell states must be employed and only N Bell states should be identified after quantum swapping. Bob needs to perform at most $2N$ single-qubit operations conditioned on Alice's $2N$-bit classical message. This leads to an important conclusion that

for *any* qubit state teleportation the use and identification of Bell states and single-qubit operations are sufficient and economical. The Comparison with the newest relevant work [16] is made, the present schemes greatly increase the use efficiency of Bell-state resource and decrease the experimental realization difficulty.

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